Final Exam
Advanced Mathematics
for Economics and Finance

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Winter Term 2015/16

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General Remarks:

- There are four questions in total.
- All problems are equally weighed.
- This is an OPEN-BOOK EXAM. You are allowed to use
  - the book by Sydsæter, *Further Mathematics for Economic Analysis* or hardcopies thereof,
  - the book by Sydsæter, *Essential Mathematics for Economic Analysis* or hardcopies thereof,
  - the book by Sydsæter, *Mathematik für Wirtschaftswissenschaftler* or hardcopies thereof,
  - the book by Chiang, *Fundamental Methods of Mathematical Economics* or hardcopies thereof,
  - your hand-written lecture notes,
  - hardcopies of the lecture slides (no electronic versions on laptops or other electronic devices),
  - hardcopies of all problem sets plus solutions to the problem sets (no electronic versions on laptops or other electronic devices),
  - a non-programmable pocket calculator.
- Good luck!

Problem 1

Consider the following quadratic form

$$Q = 2x_1^2 + 4x_1x_2 - x_2^2$$

(a) Rewrite the quadratic form in matrix notation. Call the matrix A.

(b) Determine the definiteness of the quadratic form by calculating the eigenvalues of A.

(c) Are both eigenvectors mutually orthogonal?

(d) Use a different method than that in (b) to determine the definiteness of the quadratic form.

Problem 2

Let a be a parameter. Consider the following funtion

$$f(x, y) = -x^2 + 2axy - y^2 + 4x - 4y$$

(a) Find all stationary points of f depending on a.

(b) For which values of the parameter a is the function concave/convex everywhere?

(c) For the values of a found in (b) characterize whether the stationary points are minimum or maximum or saddle points.

(d) How does the optimal value for the points characterized in (c) vary if the parameter a changes?

Problem 3

Solve the following differential equation and determine the stability of the solution:

$$\ddot{x} + 8\dot{x} + 15x = 48e^t + t$$

Problem 4

Consider the following system of differential equations

$$\begin{aligned} \dot{x} &= 2x - x^2 - y + 2 \\ \dot{y} &= x - y \end{aligned}$$

(a) Find all equilibrium points of the system.

(b) Determine for each of the equilibrium points whether it is locally asymptotically stable, a saddle point, or neither.

(c) Draw a phase diagram for the system in the first quadrant (i.e. for $x > 0, y > 0$). Explain explicitly how the arrows are determined.

Sketch of Solutions: Final Exam
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Problem 1

(a)

$$Q = 2x_1^2 + 4x_1x_2 - x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(b) Eigenvalues of A are $\lambda_1 = 3 > 0$ and $\lambda_2 = -2 < 0$ implying that Q is indefinite.

(c) Eigenvector corresponding to the eigenvalues are given by

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Yes, both eigenvectors are mutually orthogonal, as scalar product is zero.

(d) Determine the leading principal minors of A which are $D_1 = 2 > 0$ and $D_2 = -6 < 0$. implying that Q is indefinite.

Problem 2

Let a be a parameter. Consider the following funtion

$$f(x, y) = -x^2 + 2axy - y^2 + 4x - 4y$$

(a) Stationary points of f satisfy

$$\begin{aligned} f'_x(x, y) &= -2x + 2ay + 4 = 0 \\ f'_y(x, y) &= 2ax - 2y - 4 = 0 \end{aligned}$$

Solving this system for x and y yields

$$x^* = \frac{2}{1+a} \qquad \text{and} \qquad y^* = \frac{-2}{1+a}$$

(b) To determine concavity/convexity of the function f compute the Hessian matrix:

$$H = \begin{pmatrix} -2 & 2a \\ 2a & -2 \end{pmatrix}$$

The function is concave iff $f''_{xx} = -2 < 0$ and $\det H \geq 0$. The last condition is satisfied iff $-1 \leq a \leq 1$.

(c) For $-1 \leq a \leq 1$ the function is concave. In that case the function has got a maximum at (x^*, y^*).

(d) To determine how the optimal value of f changes if the parameter a changes use the envelope theorem:

$$\frac{\partial f^*(a)}{\partial a} = \left[\frac{\partial f(x,y,a)}{\partial a}\right]_{x=x^*, y=y^*} = [2xy]_{x=x^*, y=y^*} = \frac{-8}{(1+a)^2} < 0$$

So, if $a \uparrow$, then $f^*(a) \downarrow$.

Problem 3

A solution of the differential equation

$$\ddot{x} + 8\dot{x} + 15x = 48e^t + t$$

is given by first solving the homogeneous equation

$$\ddot{x} + 8\dot{x} + 15x = 0$$

Characteristic roots are given by $r_1 = -5$ and $r_2 = -3$. Thus a solution is given by

$$x = Ae^{-5t} + Be^{-3t}$$

Second, to find a particular solution of the nonhomogeneous equation use the method of undetermined coefficients. Guess a solution

$$u = Ce^t + Dt + E$$

plug it into the DE and determine the coefficients. This implies $C = 2, D = 1/15$, and $E = -8/225$.

Finally, the solution is given by

$$x = Ae^{-5t} + Be^{-3t} + 2e^t + \frac{1}{15}t - \frac{8}{225}$$

Problem 4

(a) Equilibrium points of the system

$$\begin{aligned}\dot{x} &= 2x - x^2 - y + 2 = f(x, y)\\ \dot{y} &= x - y = g(x, y)\end{aligned}$$

are given by $(-1, -1)$ and $(2, 2)$.

(b) To determine the stability of the equilibrium points compute the Jacobian matrix of the system

$$J(x, y) = \begin{pmatrix} f'_x & f'_y \\ g'_x & g'_y \end{pmatrix} = \begin{pmatrix} 2 - 2x & -1 \\ 1 & -1 \end{pmatrix}$$

Now $J(-1, -1)$ yields a negative determinant, implying that $(-1, -1)$ is a saddle point, whereas $J(2, 2)$ has a negative trace and positive determinant implying that $(2, 2)$ is stable.

(c)

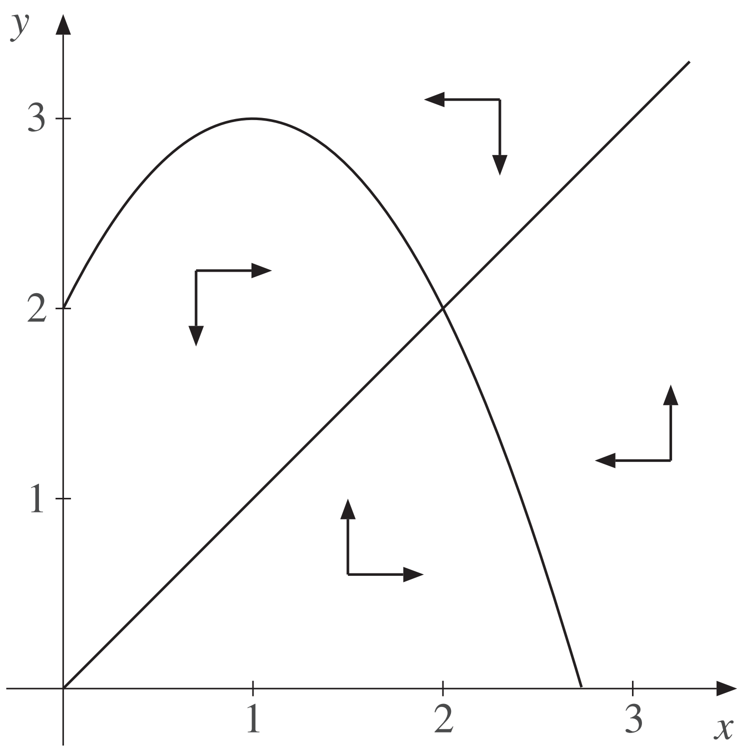

Figure 1: Phase diagram