Final Exam
Dynamic Macroeconomics

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General Remarks:

- There are four questions in total.
- All problems are equally weighed.
- This is an OPEN-BOOK EXAM. You are allowed to use
  - the book by Sydsæter, *Further Mathematics for Economic Analysis* or hardcopies thereof,
  - the book by Wickens, *Macroeconomic Theory* or hardcopies thereof,
  - your hand-written lecture notes,
  - hardcopies of the lecture slides (no electronic versions on laptops or other electronic devices),
  - hardcopies of all problem sets plus solutions to the problem sets (no electronic versions on laptops or other electronic devices),
  - a non-programmable pocket calculator.

Good luck!

Problem 1

Consider the basic centrally-planned model as discussed in class:

$$\begin{aligned} y_t &= c_t + i_t \\ \Delta k_{t+1} &= i_t - \delta k_t \\ y_t &= F(k_t) \end{aligned}$$

where the notation is as in the lecture.
The representative agent wants to maximize utility derived from life-time consumption

$$V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$$

where $0 < \beta < 1$ is the discount factor and $U(c_t)$ is the instantaneous utility function with the usual properties.

(a) Derive the consumption Euler equation of the problem assuming that utility can be specified by $U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$, and the production function is given by $F(k_t) = k_t^{\alpha}$.

(b) Assume that the central planner imposes a constant proportional tax τ on consumption. These taxes are totally destroyed by the central planner (thus there are neither invested nor given back as transfers to the household.) How does the Euler equation and the resource constraint change? Are these taxes distortionary? Discuss.

Problem 2

Consider the following system of difference equations:

$$\begin{aligned} x_{t+1} &= 4y_t \\ y_{t+1} &= 4x_t - 6y_t \end{aligned}$$

where $t = 0, 1, 2, \ldots$. Initial conditions are given by $x_0 = 1, y_0 = 0$.

Rewrite the system in matrix form:

$$X_{t+1} = AX_t \qquad \text{for} \quad X_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

and solve the system by decoupling the system.

Problem 3

Consider a firm transforming a certain substance form an initial state A into a terminal state Z through a five-stage production process. At each stage, the firm can choose among several possible subprocesses, each with a different cost. The different stages are plotted in the following figure. The costs c of each subprocess are given by the numbers in circles ©. The firm wants to select a sequence of subprocesses through the five stages in order to minimize total cost.

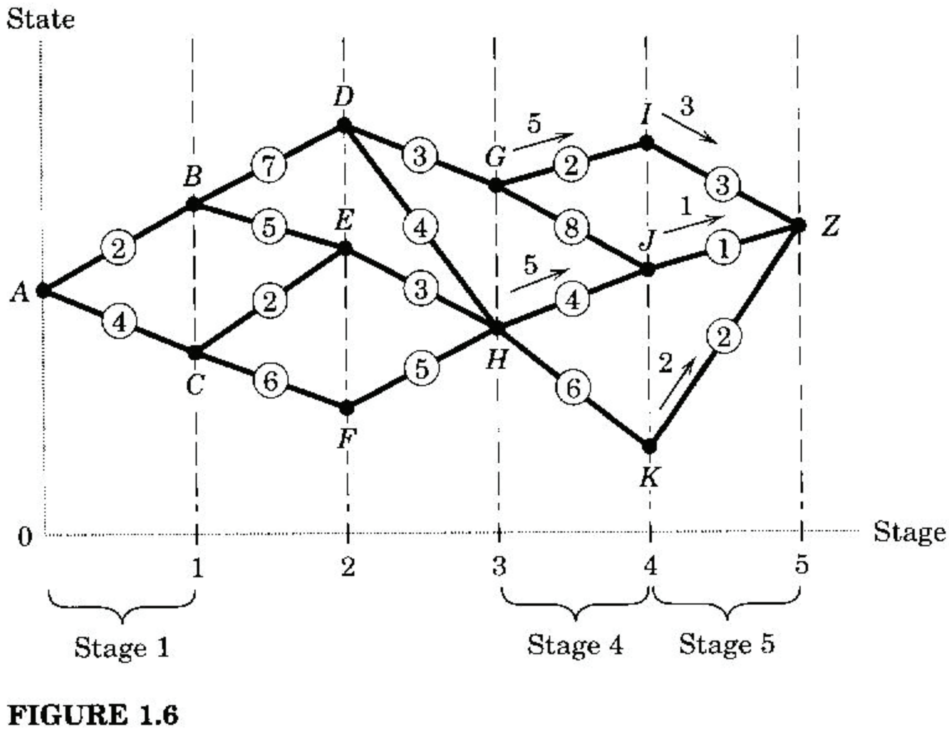

Figure 1: Multi-Stage decision problem

(a) Use this example to explain the key ideas of dynamic programming.

(b) Explain how to solve the problem. What is an appropriate value function?

(c) Which is the optimal sequence?

Problem 4

Consider the discrete time neoclassical macro model where the representative household solves the following constraint optimization problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$
$$\text{s.t.} \quad k_{t+1} = f(k_t, n_t) + (1-\delta)k_t - c_t$$

Here, u denotes (dis-)utility derived from consumption c_t and labor n_t. k_t is the capital stock and β a discount factor. Output is produced with the help of capital k_t and labor n_t and can be described by a production function f.

The first order necessary conditions of this constrained optimization problem can be summarized by the following equations:

$$u_c'(c_t, n_t) = \beta u_c'(c_{t+1}, n_{t+1}) \left[f_k'(k_{t+1}, n_{t+1}) + (1-\delta) \right] \tag{1}$$
$$u_n'(c_t, n_t) = u_c'(c_t, n_t) f_n'(k_t, n_t) \tag{2}$$
$$f(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t \tag{3}$$

(a) Assume that the production function f can be specified by $f(k_t, n_t) = A k_t^{1-\alpha} n_t^{\alpha}$ and that utility is given by $u(c_t, n_t) = \ln c_t + \theta \ln(1-n_t)$. Rewrite the first order necessary conditions (1) to (3) using these specific functions.

(b) Which conditions must hold in steady state? (Do not solve for the steady state values of c, n, and k, just state the equilibrium equations in steady state. You will need this for the next question.)

(c) Linearize both equations (1) and (2) given the specific functions around the steady state. Write down the linearized equation for variables $\hat{x}_t$ which are percentage deviations of the variable from its steady state: $\hat{x}_t = (x_t - \bar{x})/\bar{x}$.

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Sketch of Solutions

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Problem 1

(a) The Euler equation is derived by maximizing utility given the economy's resource constraint

$$F(k_t) = c_t + \Delta k_{t+1} + \delta k_t.$$

Setting up the Lagrangian (with λ_{t+s} being the Lagrange multiplier s periods ahead)

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \beta^s \, U(c_{t+s}) + \lambda_{t+s} \left[F(k_{t+s}) - c_{t+s} - k_{t+s+1} + (1-\delta) k_{t+s} \right]$$

and combining the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0 \qquad s \geq 0 \\ \frac{\partial \mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s} \left[F'(k_{t+s}) + 1 - \delta \right] - \lambda_{t+s-1} = 0 \qquad s > 0 \end{aligned}$$

yields for $s = 1$ the Euler equation

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} \left[F'(k_{t+1}) + 1 - \delta \right] = 1 \tag{1}$$

With the given assumptions about utility and the production function the Euler equation (1) gets

$$\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta \right] = 1$$

(b) When the central planner imposes a proportional tax on consumption the resource constraint gets

$$F(k_t) = (1+\tau)c_t + \Delta k_{t+1} + \delta k_t$$

Using this as a constraint in the maximization problem the Euler equation gets

$$\begin{aligned} \frac{U'(c_t)}{1+\tau} &= \frac{\beta U'(c_{t+1})}{1+\tau}\left[F'(k_{t+1}) + 1 - \delta\right] \qquad (2) \\ \iff \quad U'(c_t) &= \beta U'(c_{t+1})\left[F'(k_{t+1}) + 1 - \delta\right] \end{aligned}$$

Thus, in the Ramsey model a constant tax rate τ on consumption is not distortionary.

Problem 2

Rewriting the system

$$\begin{aligned} x_{t+1} &= 4y_t \\ y_{t+1} &= 4x_t - 6y_t \end{aligned}$$

in matrix form leads to

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Eigenvalues of the coefficient matrix A are given by μ_1=-8 and $\mu_2 = 2$. The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Define the matrix Q by

$$Q = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{aligned} \Longrightarrow \quad Q^{-1}AQ &= \begin{pmatrix} -8 & 0 \\ 0 & 2 \end{pmatrix} \equiv M \\ \iff \quad A &= QMQ^{-1} \end{aligned}$$

Use this last result to solve the matrix equation. By backwards substitution we get:

$$\begin{aligned} X_t &= A^t X_0 = QM^tQ^{-1}X_0 = Q\begin{pmatrix} (-8)^t & 0 \\ 0 & 2^t \end{pmatrix} Q^{-1}X_0 \\ &= \frac{1}{5}\begin{pmatrix} (-8)^t + 2^{t+2} \\ -2(-8)^t + 2^{t+1} \end{pmatrix} \end{aligned}$$

Problem 3

(a) The key ideas of dynamic programming is to construct an optimal value function V^* which helps to construct an optimal policy function. The original problem is solved by embedding it (this single problem) in a family of problems which are solved successively. In the example: the firm wants to find the optimal (i.e. cost-minimizing) path from the initial state A to the final state Z. Instead of finding this path we embed the problem into the family of problems of finding the optimal path from each stage (i.e. form each point to Z). Every component problem has a unique optimal path.

(b) The embedding described in (a) helps to find an iteration to solve the problem backwards. An appropriate value function is given by minimal costs. For the final stage (which takes over the role of time here) this is given by $V^* = V^*(i) =$ minimal costs going from i to Z, for $i = I, J, K$. So, the cost-minimizing path for stage 5 is the path $J - Z$. Taking this solution into account we solve the problem at stage 4: Find least-costs values going from G or H to Z taking into account that we have found out that $V^*(J)$ is minimal. This which leads to the optimal path $H - J - Z$. Thus, we move backwards, till we have found the optimal path from A to Z.

(c) The optimal sequence is given by the path $A - C - E - H - J - Z$.

Problem 4

(a) With the given specification of the production function f the utility function u the first order necessary conditions get

$$\begin{aligned} \frac{1}{c_t} &= \beta \frac{1}{c_t} \left[(1-\alpha) A k_{t+1}^{-\alpha} n_{t+1}^{\alpha} + (1-\delta) \right] && (3) \\ \frac{-\theta}{1-n_t} &= \frac{1}{c_t} \alpha A k_t^{1-\alpha} n_t^{\alpha-1} && (4) \\ A k_t^{1-\alpha} n_t^{\alpha} &= c_t + k_{t+1} - (1-\delta) k_t && (5) \end{aligned}$$

(b) In steady state the equations (3) - (5) hold for variables $c_t = c, k_t = k$, and $n_t = n$. Equation (3) and (5) simplify.

$$\begin{aligned} 1 &= \beta \left[(1-\alpha) A k^{-\alpha} n^{\alpha} + (1-\delta) \right] \\ \frac{-\theta}{1-n} &= \frac{1}{c} \alpha A k^{1-\alpha} n^{\alpha-1} \\ A k^{1-\alpha} n^{\alpha} &= c + \delta k \end{aligned}$$

(c) While linearizing both equations (3) and (4) you have to take into account that the utility function depends on consumption c_t and labor n_t and the production function depends on capital k_t and labor n_t. So you have to use the Taylor rule for bivariate functions or you have to use the product rule when taking total derivatives. No matter which method you apply you will get linear equations in c_t, c_{t+1}, k_{t+1} plus n_t, n_{t+1}. Moreover, take into account the steady state relationships derived in (b). Thus many coefficients of the linearized equation simplify. To sum up, the linearized versions of (3) and (4) are given by:

$$\begin{aligned}
\hat{c}_t &= \hat{c}_{t+1} + \beta(1-\alpha)\alpha A k^{-\alpha} n^{\alpha} \hat{k}_{t+1} - \beta(1-\alpha)\alpha A k^{-\alpha} n^{\alpha} \hat{n}_{t+1} \\
\frac{n}{1-n}\hat{n}_t &= -\hat{c}_t + (1-\alpha)\hat{k}_t + (\alpha-1)\hat{n}_t
\end{aligned}$$